Exact Solutions of the One-Dimensional Quintic Complex Ginzburg-Landau Equation

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Abstract

Exact solitary wave solutions of the one-dimensional quintic complex Ginzburg-Landau equation are obtained using a method derived from the Painlevé test for integrability. These solutions are expressed in terms of hyperbolic functions, and include the pulses and fronts found by van Saarloos and Hohenberg. We also find previously unknown sources and sinks. The emphasis is put on the systematic character of the method which breaks away from approaches involving somewhat ad hoc Ansätze.

1 Introduction

A number of non-integrable, non-linear, dissipative partial differential equations (PDEs) are known to display a wide variety of complex behavior, where the global time evolution is often governed by the dynamics of spatially localized structures. For example, in defect-mediated turbulence [2], the disordered creation, motion and annihilation of topological defects play a prominent role. In some cases of spatiotemporal intermittency, well-defined localized objects "carry" the disorder and act as spatial delimiters of laminar regions [3]. This was shown in particular for the Nozaki-Bekki family of exact solutions of the one-dimensional supercritical complex Ginzburg-Landau equation (see below, Eq. (4)) [4]. More strongly disordered regimes of simple PDEs like the Kuramoto-Sivashinsky equation, in which localized objects do not appear in an obvious manner, have been argued to be well-described as a "gas" of interacting pulses which are themselves exact solutions of the governing equation [5].

All these objects are, to a variable extent, reminiscent of the solitons of completely integrable equations. They can be thought of (at least on a qualitative level) as the dissipative counterparts of invariant tori in chaotic Hamiltonian systems, in the sense that they represent the preserved part of the rich mathematical structure of nearby integrable systems. Methods developed to investigate integrability of differential equations can therefore be expected to shed new light on spatiotemporally chaotic dynamics [6].

Following this line of thought, we present new exact particular solutions for the quintic Complex Ginzburg-Landau (CGL) equation in one spatial dimension, using techniques derived from the Painlevé test for integrability [7]. The quintic CGL equation reads:

$$\frac{\partial A}{\partial t} = \varepsilon A + (b_1 + ic_1)\frac{\partial^2 A}{\partial x^2} - (b_3 - ic_3)|A|^2 A - (b_5 - ic_5)|A|^4 A, \quad (1)$$

where ε, b_1, c_1, b_3, c_3, b_5, c_5, are real constants and the field $A(x,t)$ is complex.

Eq. (1) is a one-dimensional model of the large-scale behavior of many nonequilibrium pattern-forming systems [8]. When the real parameters b_3 and b_5 are respectively negative and positive, Eq. (1) accounts for the slow modulations in space and time of an oscillatory mode close to a subcritical

Hopf bifurcation. The discontinuous character of this symmetry-breaking bifurcation is responsible for the occurrence of metastable states separated by fronts [9]. Pulse solutions have also been argued to exist and play an important dynamical role [10]. Examples of relevant experimental contexts include binary fluid convection [11] and Taylor-Couette flow between counter-rotating cylinders [12].

We build on the work of W. van Saarloos and P.C. Hohenberg [1], who recently reviewed the properties of the solutions of Eq. (1) which they called *coherent structures* in order to emphasize their strong, usually exponential, spatial localization. In order to attempt to stop the proliferation of notations, we have chosen to use their notation throughout this paper, as well as their vocabulary to distinguish among the various types of coherent structures.

Exact, analytical solutions of the CGL equation are scarce [1, 13, 14], and in any case limited to the uniformly propagating case, *i.e.* solutions of the form:

$$A(x,t) = e^{-i\omega t} \, \hat{A}(\xi = x - vt). \qquad (2)$$

The original PDE, depending upon (x, t), is thus reduced to a second-order ordinary differential equation (ODE) in the $\xi = x - vt$ independent variable, where v is a constant velocity, and " $'$ " stands for differentiation with respect to ξ:

$$-v\hat{A}' = (\varepsilon + i\omega)\hat{A} + (b_1 + ic_1)\hat{A}'' - (b_3 - ic_3)|\hat{A}|^2\hat{A} - (b_5 - ic_5)|\hat{A}|^4\hat{A}. \quad (3)$$

Eq. (3) possesses two types of fixed point: linear fixed points which correspond to a trivial vacuum state, $A(x,t) = 0$, and non-linear fixed points which correspond to plane waves:

$$A(x,t) = a_N e^{-i\omega_N t + iq_N x}.$$

The next step consists in looking for connections between any two of these elementary objects in the phase space of Eq. (3). These connecting objects, the coherent structures mentioned above, are classified as follows:

- pulses, *i.e.* homoclinic orbits between two vacuum states (linear fixed points) at $\xi = \pm\infty$,

- fronts, *i.e.* heteroclinic orbits between a vacuum state at $\pm\infty$ and a plane wave (non-linear fixed point) at $\mp\infty$,

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- sources, *i.e.* heteroclinic orbits between two outgoing waves at $\pm\infty$,

- sinks, *i.e.* heteroclinic orbits between two incoming waves at $\pm\infty$.

Sources and sinks can be distinguished from each other by checking the signs of the group velocities of the asymptotic plane waves at $\pm\infty$. The multiplicity of these coherent structures admits an upper bound, determined by the dimensionality of the connecting manifolds, flowing from the unstable eigendirection(s) of one fixed point into the stable eigendirection(s) of another fixed point.

Exact solutions to Eq. (3) were obtained in [1] by using an ad hoc, reduction of order Ansatz, where the first-order derivatives of the phase and amplitude of $A(\xi)$ are given *a priori* expressions. Van Saarloos and Hohenberg found exact pulses and fronts, but did not mention the existence of any exact source or sink for the quintic equation, although such solutions are allowed by the counting arguments. Finally, they gave numerical evidence of the important role played by these special, highly non-generic solutions, which were shown to be "dynamically selected" in certain regions of parameter space for sufficiently localized initial conditions.

The purpose of this paper is to show that a coherent mathematical framework can foster a better understanding of these exact solutions and of the precise reasons for their functional form. The emphasis is laid upon a *local* study of the *analytical* structure of possible solutions in the complex plane, as opposed to the more geometrical methods referred to above. Similar arguments were recently used by Conte and Musette in the case of the cubic CGL equation [15], for which the quintic term vanishes:

$$\frac{\partial A}{\partial t} = \varepsilon A + (b_1 + ic_1)\frac{\partial^2 A}{\partial x^2} - (b_3 - ic_3)|A|^2 A. \qquad (4)$$

All the known solutions of (4) were naturally retrieved in [15], including the one-parameter family of sources originally due to Nozaki and Bekki [16].

In addition to the known pulses and fronts of [1], we find for the quintic CGL equation a new set of sinks and sources, whose existence is restricted to a low co-dimension subspace of the full $(\varepsilon, b_1, c_1, b_3, c_3, b_5, c_5)$ parameter space. These three types of solution locally obey the same singularity structure, and use hyperbolic functions as the elementary units from which their

global functional form is built. However, their respective multiplicities do not reach the upper bounds set in [1].

2 Methodology

2.1 The Painlevé test for integrability

Our guideline will be integrability in the sense of Painlevé: an ODE will be called integrable if its general solution is free from *movable critical* points.

Let us first define these adjectives. A critical point (of a complex-valued application) is a point around which several determinations of the application occur. Examples include algebraic and logarithmic branch points.

A movable singular point (of a solution of a DE) is a singular point whose location in the complex plane is not determined by the coefficients of the DE. This location can only be obtained from the initial conditions of the differential problem, *i.e.* from integration constants. The simplest example is provided by the Bernoulli equation:

$$u'(x) = -u(x)^2,$$

whose general solution

$$u(x) = \frac{1}{x - x_0}$$

admits a movable simple pole at x_0. Conversely, singular points whose location depends only upon the coefficients of the DE are called fixed. Linear DE's only admit fixed critical points.

In this context, integrability is intimately connected with single-valuedness: integrating a differential equation is ultimately equivalent to expressing its solution in terms of functions, *i.e.* single-valued applications of $\mathbf{C}$ into $\mathbf{C}$. Multi-valuedness, expressed through the occurrence of critical points, can be easily dispensed with when the critical points are fixed, for instance by removing from the domain of the solution a line in $\mathbf{C}$ between the critical point and a point at infinity. On the other hand, movable critical points are sources of persistent multi-valuedness, and therefore preclude integrability.

Implicit in this scheme is the necessity of extending the domain of independent variables to the complex plane. Although unphysical at first sight,

this prerequisite is simply analogous to solving real algebraic equations for complex unknowns.

The strength of the Painlevé test for an ODE

$$E\left[\frac{\mathrm{d}}{\mathrm{d}x},\ u(x)\right] = 0 \tag{5}$$

lies in its easy, algorithmic implementability. Its main requirement is the existence of all possible solutions $u(x)$ of Eq. (5) expressed as a Laurent expansion in a neighborhood of a movable singularity x_0:

$$u(x) = \chi(x)^{-\alpha} \sum_{j=0}^{\infty} u_j\ \chi(x)^j, \tag{6}$$

where α is the leading-order exponent, $\chi(x)$ the expansion variable, and $\{u_j,\ j \geq 0\}$ a set of constant coefficients. Following the invariant formulation of Conte [18], we distinguish here the expansion variable $\chi(x)$ from the singular manifold of Weiss, Tabor, and Carnevale [7], and only require $\chi(x)$ to behave as a simple zero near x_0: $\chi(x) \sim x - x_0$. This expansion is substituted into $E[\mathrm{d}/\mathrm{d}x, u(x)]$. The u_j's are determined from recursion relations that develop when the coefficients at each order of χ are required to vanish. The leading-order exponent α is determined by equating the exponents of the dominant order terms in the DE (5).

Necessary conditions for an ODE to pass the Painlevé test are:

1. the leading order α is an integer,

2. the recursion relation for the coefficients u_j can be consistently solved to any order,

and possibly some other conditions not detailed here [19]. This procedure checks that the Laurent-type expansion for $u(x)$ (Eq. (6)) is both consistent and free from logarithmic branch points.

2.2   Painlevé analysis for nonintegrable equations

The general solution of nonintegrable equations will fail the Painlevé test at one of these two steps. However, this does not forbid the existence of particular solutions, provided that they respect the singularity structure derived from the leading-order analysis.

The next step consists in determining how the local, analytical structure valid in a neighborhood of a singular point x_0 can be taken into account to yield global results, namely expressions of $u(x)$ valid for **C** as a whole. Once again, the guideline is provided by one of the early results of Painlevé on integrability [17]: the solutions of all known integrable non-linear ODEs of order at most two and degree one in $u''(x)$ can be expressed as linear combinations of logarithmic derivatives of entire functions, whose coefficients are entire functions [21]. We mention here as an example the case of (P2), one of the six integrable second-order equations of Painlevé:

$$u'' = 2u^3 + xu + a$$

where a is a constant coefficient. Its general solution can be expressed as:

$$u(x) = \partial_x \text{Log} \psi_1 - \partial_x \text{Log} \psi_2,$$

ψ_1 and ψ_2 being two entire functions.

This result should not come as a surprise, since $\partial_x \text{Log} \psi$ is by construction single-valued, and behaves like a pole. In this respect, the integrable (thus single-valued) part of nonintegrable equations is naturally expected to be expressible in terms of logarithmic derivatives of entire functions.

We now turn to the definition of the class of possible solutions we consider. Arguments will remain mostly heuristic, although our Ansatz can be derived within a rigorous mathematical setting, taking into account the inherent invariance of Painlevé analysis under the group of homographic transformations. For more details, we refer the more mathematically-oriented reader to the articles [15] and [18] and to the lecture notes [21].

2.3 Ansatz for the quintic CGL equation

Leading-order analysis for Eq. (3) is achieved by balancing the highest-order derivative with the strongest nonlinearity. $\hat{A}(\xi)$ being a complex field, this must be done by writing two complex conjugate equations for $\hat{A}(\xi)$ and $\hat{B}(\xi) = \hat{A}^*(\xi)$, where "*" denotes complex conjugation. The fields $\hat{A}$ and $\hat{B}$ are now formally considered as independent variables, and obey:

$$\begin{aligned}
(b_1 + ic_1)\hat{A}'' &\sim (b_5 - ic_5)\hat{A}^3\hat{B}^2, \\
(b_1 - ic_1)\hat{B}'' &\sim (b_5 + ic_5)\hat{A}^2\hat{B}^3.
\end{aligned} \tag{7}$$

Using $\chi(\xi) \sim \xi - \xi_0$ and $\chi'(\xi) \sim 1$, and feeding the leading-order Ansatz $\hat{A} \sim A_0 \chi^\alpha$, $\hat{B} \sim B_0 \chi^\beta$ into Eqs. (7) leads to:

$$
\begin{aligned}
\hat{A} &\sim A_0 \; \chi^{-\frac{1}{2}+i\alpha_0}, \\
\hat{B} &\sim B_0 \; \chi^{-\frac{1}{2}-i\alpha_0},
\end{aligned}
\tag{8}
$$

where A_0, B_0 and α_0 are solutions of the equations:

$$
\begin{aligned}
\mathcal{I} \; \alpha_0^2 + 2\mathcal{R} \; \alpha_0 - \tfrac{3}{4}\mathcal{I} &= 0, \\
(A_0 B_0)^2 &= 2 \; \tfrac{b_1^2 + c_1^2}{\mathcal{I}} \; \alpha_0,
\end{aligned}
$$

and the intermediate variables $\mathcal{R}$ and $\mathcal{I}$ are defined as:

$$
\begin{aligned}
\mathcal{R} &= \mathrm{Re} \left[(b_1 + ic_1)(b_5 + ic_5) \right] &= b_1 b_5 - c_1 c_5, \\
\mathcal{I} &= \mathrm{Im} \left[(b_1 + ic_1)(b_5 + ic_5) \right] &= b_1 c_5 + c_1 b_5.
\end{aligned}
$$

Without loss of generality, we assume that $A_0 = B_0$ are real constants in the rest of this paper.

We consider here the non-degenerate case $\mathcal{I} \neq 0$, where:

$$
\begin{aligned}
\alpha_0 &= -\tfrac{\mathcal{R}}{\mathcal{I}} \pm \sqrt{\tfrac{3}{4} + \left(\tfrac{\mathcal{R}}{\mathcal{I}}\right)^2} \neq 0, \\
A_0^4 &= 2 \; \tfrac{b_1^2 + c_1^2}{\mathcal{I}} \; \alpha_0.
\end{aligned}
\tag{9}
$$

The degenerate case $\mathcal{I} = 0$, $\alpha_0 = 0$ is treated in the Appendix. It should be noted that all results we present (including the Appendix) respect the leading-order balance (7), and are thus valid only when the coefficients of the quintic term and of the highest order derivative are both non-zero:

$$
\begin{cases}
b_1 + ic_1 &\neq \; 0, \\
b_5 + ic_5 &\neq \; 0.
\end{cases}
\tag{10}
$$

The leading-order exponent $-\frac{1}{2} + i\alpha_0$ is not an integer: the nonintegrable complex Ginzburg-Landau equation has already failed the Painlevé test. As expected, its general solution cannot be expressed in terms of elementary functions. However, partial integrability remains possible, in so far as one looks for solutions exhibiting minimal - but necessary - multi-valuedness, including movable algebraic and logarithmic branch points. This requirement is clearly fulfilled by the following rewriting of Eq. (8):

$$
A(\xi) = A_0 \; e^{-i\omega t} \; R(\xi)^{\frac{1}{2}} \; e^{i\alpha_0 \Theta(\xi)}.
\tag{11}
$$

We introduced here a generalized amplitude $R(\xi)$ and a generalized phase $\Theta(\xi)$, *a priori* complex-valued. The amplitude $R(\xi)$ is assumed to have at worst the singular behavior of a pole, $\Theta(\xi)$ that of a logarithmic branch point.

The global structure of possible solutions is introduced as follows. To a given leading-order exponent α_0 correspond four distinct values of A_0, related through a phase shift of $\frac{\pi}{2}$. Four families of entire functions ψ_i, $i = 1, ..., 4$ must therefore be introduced in the expression of $R(\xi)$:

$$R(\xi) = r_0(\xi) + \sum_{i=1}^{4} r_i(\xi) \; \partial_\xi \mathrm{Log}\psi_i. \tag{12}$$

The coefficients r_i, $i = 0, ..., 4$ are assumed to be entire functions of ξ, thus ensuring a pole behavior for $R(\xi)$ in the vicinity of any of the movable zeroes of the ψ_i's. For simplicity, we will restrict our Ansatz to the case where:

- only two families ψ_1 and ψ_2 are used, and

- the coefficients r_0, r_1 and r_2 are real constants,

thus keeping the number of unknowns at a tractable level with respect to the number of equations. Leading-order analysis leads to $r_1 = \pm r_2 = \pm 1$, when conducted according to Eq. (12) with r_0, r_1, r_2 real constants, and $r_3 = r_4 = 0$. Up to a constant phase shift, we can fix $r_1 = +1$, $r_2 = \pm 1$. Our Ansatz for $R(\xi)$ reads:

$$R(\xi) = r_0 + \partial_\xi \mathrm{Log}\psi_1 \pm \partial_\xi \mathrm{Log}\psi_2. \tag{13}$$

Let us now turn to the Ansatz for $\Theta(\xi)$. Only one family is necessary here, which we denote ψ_1. Since the complex Ginzburg-Landau equation is invariant under an homogeneous phase translation $A \to A \, e^{i\phi}$, Θ only contributes through its gradient $\partial_\xi \Theta$. We only need to define an expression for $\partial_\xi \Theta$, with, again, a pole singular behavior at worst. This expression reads:

$$\partial_\xi \Theta(\xi) = \theta_0 + \partial_\xi \mathrm{Log}\psi_1, \tag{14}$$

where a constant coefficient θ_0 was introduced.

The entire functions ψ_1 and ψ_2 are next defined as solutions of integrable differential equations. For simplicity, we assume that the ψ_i's are solutions to the second-order linear ODE:

$$\frac{\mathrm{d}^2\psi_i}{\mathrm{d}\xi^2} = \frac{k^2}{4} \; \psi_i. \tag{15}$$

Linear first-order differential equations have been excluded. Their solutions have constant logarithmic derivatives, leading to trivial expressions of $R(\xi)$ and $\Theta(\xi)$ which correspond to the fixed-point solutions of Eqs. (3). Other choices of the defining ODE are possible in principle, but were at first discarded, again for reasons of simplicity. This helped in keeping the algebraic manipulations needed later at a tractable level.

The general solution to Eq. (15) reads:

$$\psi(\xi) = \psi_0 \, \cosh \frac{k}{2}(\xi - \xi_0), \tag{16}$$

where ψ_0 and ξ_0 are the two integration constants. The value of ψ_0 can be set to 1, since only the ratio $\partial_\xi \psi_i / \psi_i$ contributes. We define the two families as two independent solutions to Eq. (16) separated by a constant phase shift denoted ka:

$$\begin{aligned}
\psi_1(\xi) &= \cosh \tfrac{k}{2}(\xi - \xi_0 + a), \\
\psi_2(\xi) &= \cosh \tfrac{k}{2}(\xi - \xi_0 - a).
\end{aligned} \tag{17}$$

Elementary manipulations then show that [15]:

$$\partial_\xi \text{Log} \psi_1 \;+\; \partial_\xi \text{Log} \psi_2 \;=\; \frac{k \, \sinh k(\xi - \xi_0)}{\cosh k(\xi - \xi_0) \;+\; \cosh(ka)}, \tag{18}$$

$$\partial_\xi \text{Log} \psi_1 \;-\; \partial_\xi \text{Log} \psi_2 \;=\; \frac{k \, \sinh(ka)}{\cosh k(\xi - \xi_0) \;+\; \cosh(ka)}. \tag{19}$$

Our Ansatz (17) can be expected to yield all possible solutions to the complex Ginzburg-Landau equation involving hyperbolic functions or, up to linear combinations, exponentials. Hyperbolic tangents can be obtained from Eq. (18) when $\cosh(ka) = 0$, $i.e.$ $ka = i\frac{\pi}{2}$, hyperbolic secants from Eq. (19) when $\cosh(ka) = 0$.

Eq. (18) shows that two families having identical coefficients are in practice equivalent to one family, up to dividing the wavenumber k by 2 and to setting $ka = i\frac{\pi}{2}$. This allows a slight modification of the equation defining $\Theta(\xi)$, formally using the two families ψ_1 and ψ_2. The complete Ansatz for $A(\xi)$ now reads:

$$\left\{ \begin{aligned}
A(\xi) &= A_0 \, e^{-i\omega t} \, R(\xi)^{\frac{1}{2}} \, e^{i\alpha_0 \Theta(\xi)}, \\
R &= r_0 \;+\; \partial_\xi \text{Log} \psi_1 \;\pm\; \partial_\xi \text{Log} \psi_2, \\
\partial_\xi \Theta &= \theta_0 \;+\; \partial_\xi \text{Log} \psi_1 \;+\; \partial_\xi \text{Log} \psi_2, \\
\partial_{\xi\xi} \psi_i &= \tfrac{k^2}{4} \, \psi_i, \quad i = 1, 2.
\end{aligned} \right. \tag{20}$$

Albeit very restrictive, this Ansatz suffices to retrieve all the exact solutions quoted in [1].

2.4 Computational aspects

From now on, the constant coefficients θ_0, ω, c will be real numbers, in order to avoid unbounded solutions. The wavenumber k is also assumed to be real. Possible periodic solutions are thus discarded, since they lack the required spatial localization. On the other hand, taking $r_0 \in \mathbf{C}$ would not restrict the generality of Ansatz (20). However,we checked that the quintic equation, unlike the cubic one (see [15]), does not admit any solutions respecting the less restrictive hypothesis $r_0 = x_0 + i\, y_0$; x_0, $y_0 \in \mathbf{R}$; $x_0 y_0 \neq 0$.

The basic principle of our resolution is to turn the differential equation we want to solve into a much simpler, purely algebraic problem. This is made possible by the analytic considerations of the previous section: the spatial structure of solutions to Eq. (3) is supposed to be fully contained in the elementary logarithmic derivatives $\partial_\xi \mathrm{Log}\psi_i$, whose functional form will never be made explicit in our computations. The sometimes intricate algebraic manipulations can then be solved quite easily with the help of any symbolic mathematics package, such as Mathematica [22], or AMP [23].

We first substitute our general Ansatz (Eqs. (20)) into Eq. (3) and successively eliminate all derivatives of ψ_i of order greater than or equal to two by using:

$$\frac{\mathrm{d}^2 \psi_i}{\mathrm{d}\xi^2} = \frac{k^2}{4}\, \psi_i.$$

Eq. (3) is then equivalent to a polynomial equation in the $\partial_\xi \mathrm{Log}\psi_1$ and $\partial_\xi \mathrm{Log}\psi_2$ variables:

$$\sum_{k=0}^{4} \sum_{m+n=k} F_k \, (\partial_\xi \mathrm{Log}\psi_1)^m \, (\partial_\xi \mathrm{Log}\psi_2)^n = 0, \tag{21}$$

where the coefficients F_k depend algebraically on the parameters (ε, b_i, c_i), on the unknowns ω, k, v, a, r_0, θ_0, and on A_0 and α_0, whose values are known from Eq. (9).

A convenient way of taking the phase shift ka into account is to use a new variable μ_0, defined as $\mu_0 = \coth(ka)$. The products $(\partial_\xi \mathrm{Log}\psi_1)^m \, (\partial_\xi \mathrm{Log}\psi_2)^n$ can be recursively linearized, from $m+n=4$ to $m+n=2$, by means of the following identity:

$$\partial_\xi \mathrm{Log}\psi_1 \, \partial_\xi \mathrm{Log}\psi_2 = \frac{k^2}{4} - \mu_0 \, \frac{k}{2} \, [\, \partial_\xi \mathrm{Log}\psi_1 - \partial_\xi \mathrm{Log}\psi_2 \,]. \tag{22}$$

The coefficient μ_0 is now a real constant, in agreement with the previous assumptions. We obtain:

$$\sum_{j=1}^{4} E_{-j} \ (\partial_\xi \mathrm{Log}\psi_1)^j \ + \ E_0 \ + \ \sum_{j=1}^{4} E_j \ (\partial_\xi \mathrm{Log}\psi_2)^j \ = 0. \qquad (23)$$

Solving Eq. (23) amounts to canceling all the E_j coefficients, $j = -4, \ldots, 4$. This is done recursively, from E_{-4} and E_4 to E_0, by a triangularization technique. Parameters and unknowns are considered on an equal footing, as variables whose values are successively obtained from E_j and substituted into equations E_i, $0 \leq |i| < |j|$, thus decreasing the number of unknowns in equations of lower index. For each j, the pivoting variable was determined by singling out which variable admits the simplest expression with respect to all other variables, while excluding possibly vanishing denominators. These selected variables are:

- $a_5 = b_5 - ic_5$, obtained from E_{-4} or E_4,

- $a_3 = b_3 - ic_3$, obtained from E_{-3},

- $\hat{v} = v + 2i\alpha_0 a_1 \theta_0$, obtained from E_3, where $a_1 = b_1 + ic_1$, and

- $\hat{\varepsilon} = \varepsilon + i\theta_0\omega$, obtained from E_{-2}.

Equation E_2 is then seen to vanish, and the remaining E_{-1}, E_0 and E_1 equations depend only on the parameters b_1, c_1, the leading-order quantities A_0, α_0, and the unknowns k, μ_0 and r_0. Systematic resolution of these equations lead to the three cases detailed below, where the explicit values of the real unknowns k, v, ω, θ_0, r_0 and μ_0 are given as functions of the parameters (ε, b_i, c_i), A_0 and α_0. The leading-order quantities A_0 and α_0 are considered as parameters, since their value can be expressed as functions of b_1, c_1, b_5, and c_5 (Eq. (9)).

3 Results

We do not mention here "unphysical" solutions obtained for complex values of the parameters k, v, ω, θ_0, r_0 or μ_0, although such solutions may well become "physical" when more general hypotheses are considered. These questions have been left for future work.

3.1 Pulses

The pulse solutions given in [1] can be obtained from the following Ansatz:

$$\begin{cases} R & = & \partial_\xi \text{Log} \psi_1 - \partial_\xi \text{Log} \psi_2, \\ \partial_\xi \Theta & = & \theta_0 + \partial_\xi \text{Log} \psi_1 + \partial_\xi \text{Log} \psi_2, \end{cases}$$

where r_0 was set to 0 in order to respect the suitable asymptotic behavior:

$$\lim_{\xi \to -\infty} A(\xi) = \lim_{\xi \to +\infty} A(\xi) = 0.$$

The solution reads:

$$A(x,t) = A_0 e^{-i\omega t} e^{i\alpha_0 \theta_0 \xi} \left[\cosh k(\xi - \xi_0) + \cosh ka \right]^{i\alpha_0} \tag{24}$$
$$\times \left[\frac{k \ \sinh(ka)}{\cosh k(\xi - \xi_0) \ + \ \cosh(ka)} \right]^{\frac{1}{2}}.$$

In this particular case, triangularizing the equations E_j, $j = -4, ..., 4$ leads to $\hat{v} = 0$, or $v = -2i\alpha_0(b_1 + ic_1)\theta_0$. The velocity v being a real constant, we obtain:

$$b_1 \theta_0 = 0,$$
$$v = 2c_1 \alpha_0 \theta_0.$$

We find a set of solutions, discrete when $b_1 \neq 0$, $v = \theta_0 = 0$, parametrized by the velocity v or by θ_0 when $b_1 = 0$, in a co-dimension-one subspace of parameter-space defined by:

$$c_3 \left[b_1(1 - 2\alpha_0^2) + 3\alpha_0 c_1 \right] = b_3 \left[3\alpha_0 b_1 + c_1(2\alpha_0^2 - 1) \right].$$

The unknowns k, μ_0 and ω can be computed from:

$$k^2 = -\frac{4\varepsilon}{b_1(1 - 4\alpha_0^2) + 4\alpha_0 c_1},$$
$$k\mu_0 = \frac{-b_3 A_0^2}{b_1(1 - 2\alpha_0^2) + 3\alpha_0 c_1},$$
$$\omega = -c_1 \alpha_0 \theta_0^2 + \frac{\varepsilon}{\alpha_0} \frac{-4\alpha_0 b_1 + c_1(1 - 4\alpha_0^2)}{b_1(1 - 4\alpha_0^2) + 4\alpha_0 c_1}.$$

3.2 Fronts

The front solutions of [1] involve hyperbolic tangents. The Ansatz we use goes as follows:

$$\begin{cases} R & = & r_0 + \partial_\xi \mathrm{Log}\psi_1 + \partial_\xi \mathrm{Log}\psi_2, \\ \partial_\xi \Theta & = & \theta_0 + \partial_\xi \mathrm{Log}\psi_1 + \partial_\xi \mathrm{Log}\psi_2. \end{cases}$$

Solutions were found only when $\mu_0 = 0$. Their explicit form reads:

$$A(x,t) = A_0 e^{-i\omega t} e^{i\alpha_0\theta_0\xi} \left[\cosh k(\xi - \xi_0)\right]^{i\alpha_0} \left[k(\tanh k(\xi - \xi_0) \pm 1)\right]^{\frac{1}{2}}, \quad (25)$$

where the appropriate asymptotic behavior is obtained by setting r_0 to $\pm k$:

$$\begin{aligned} \lim_{\xi \to \mp\infty} A(\xi) &= 0, \\ \lim_{\xi \to \pm\infty} A(\xi) &= \sqrt{\pm 2k} A_0 e^{-i\omega t} e^{i\alpha_0(\theta_0 \pm k)}. \end{aligned}$$

Its variables obey different relationships according to the value of b_1:

- **First subcase**: $b_1 \neq 0$. A discrete set of solutions is found. The wavenumber k is determined by a quadratic equation:

$$a_k k^2 + b_k k + c_k = 0,$$

where:

$$\begin{cases} a_k & = & -(1 + 4\alpha_0^2)^2 \, (3b_1^2 + 4c_1^2), \\ b_k & = & \mp 2A_0^2 \, (1 + 4\alpha_0^2) \, (b_1 b_3 - 2c_1 c_3 + 2\alpha_0(b_1 c_3 + 2b_3 c_1)), \\ c_k & = & b_1 \varepsilon \, (1 + 4\alpha_0^2)^2 - A_0^4 \, (c_3^2 + 4b_3^2\alpha_0^2 - 4\alpha_0 b_3 c_3). \end{cases}$$

The unknowns θ_0, v and ω are obtained as functions of k:

$$\begin{aligned} \theta_0 &= \pm k \, (1 - 2\frac{c_1}{\alpha_0 b_1}) + A_0^2 \, \frac{c_3 - 2\alpha_0 b_3}{\alpha_0(1 + 4\alpha_0^2)b_1}, \\ v &= \mp 2\frac{b_1^2 + c_1^2}{b_1} k - 2 \, \frac{A_0^2}{b_1} \, \frac{(c_1 c_3 - b_1 b_3) + 2\alpha_0(b_1 c_3 + b_3 c_1)}{1 + 4\alpha_0^2}, \\ \omega &= \frac{1}{b_1^2 \alpha_0(1 + 4\alpha_0^2)^2} \, [\, a_\omega k^2 + b_\omega k + c_\omega \,], \end{aligned}$$

where:

$$\begin{cases} a_\omega & = & -c_1(1 + 4\alpha_0^2)^2 \, (5b_1^2 + 4c_1^2), \\ b_\omega & = & \pm \, 2A_0^2(1 + 4\alpha_0^2) \\ & & \times [(b_1^2 c_3 + 2c_1^2 c_3 - 2b_1 b_3 c_1) - 2\alpha_0 \, (b_1^2 b_3 + 2b_3 c_1^2 + 2b_1 c_1 c_3)], \\ c_\omega & = & A_0^4 \, (c_3 - 2\alpha_0 b_3) \, [(2b_1 b_3 - c_1 c_3) + 2\alpha_0(b_3 c_1 + 2b_1 c_3)]. \end{cases}$$

- **Second subcase**: $b_1 = 0$. In the non-degenerate case treated here ($\mathcal{I} = b_1 c_5 + c_1 b_5 \neq 0$), this condition implies that c_1 cannot vanish. We find a one-parameter family of solutions restricted to a co-dimension-two subspace of parameter space defined by:

$$b_1 = 0,$$
$$c_1(1 + 4\alpha_0^2)^2 \, \varepsilon = (c_3 - 2\alpha_0 b_3)(b_3 + 2\alpha_0 c_3) \, A_0^4.$$

The unknowns θ_0, ω and k are parametrized by the velocity v:

$$\theta_0 = \frac{v}{2\alpha_0 c_1} + \frac{A_0^2}{2\alpha_0 c_1} \frac{2b_3(1 - \alpha_o^2) + 5\alpha_0 c_3}{1 + 4\alpha_0^2},$$

$$\omega = -\frac{v^2}{4\alpha_0 c_1}$$
$$+ \frac{A_0^4}{4\alpha_0(1 + 4\alpha_0^2)^2 c_1} [(2b_3 + 4\alpha_0 c_3)^2 - (2\alpha_0 b_3 - c_3)^2],$$

$$k = \pm \frac{A_0^2}{2c_1} \frac{c_3 - 2\alpha_0 b_3}{1 + 4\alpha_0^2}.$$

3.3   Sources and sinks

Sources and sinks are obtained from the Ansatz:

$$\begin{cases} R &= r_0 + \partial_\xi \mathrm{Log}\psi_1 - \partial_\xi \mathrm{Log}\psi_2, \\ \partial_\xi \Theta &= \theta_0 + \partial_\xi \mathrm{Log}\psi_1 + \partial_\xi \mathrm{Log}\psi_2, \\ r_0 &\neq 0. \end{cases}$$

Solutions exist for $\mu_0 \neq 0$ only, and interpolate between two plane waves:

$$\lim_{\xi \to -\infty} A(\xi) = A_0 r_0 e^{-i\omega t} e^{i[\,\alpha_0(\theta_0 - k)\xi\,]},$$
$$\lim_{\xi \to +\infty} A(\xi) = A_0 r_0 e^{-i\omega t} e^{i[\,\alpha_0(\theta_0 + k)\xi\,]}.$$

Their expression reads:

$$A(x, t) = A_0 e^{-i\omega t} e^{i\alpha_0 \theta_0 \xi} [\,\cosh k(\xi - \xi_0) + \cosh ka\,]^{i\alpha_0} \qquad (26)$$
$$\times \left[\frac{k \,\sinh(ka)}{\cosh k(\xi - \xi_0) \,+\, \cosh(ka)} + r_0 \right]^{\frac{1}{2}}.$$

As for pulses, we obtain $\hat{v} = 0$, whence:

$$b_1 \theta_0 = 0,$$
$$v = 2c_1 \alpha_0 \theta_0.$$

We find a set of solutions, discrete when $b_1 \neq 0$, $v = \theta_0 = 0$, parametrized by the velocity v when $b_1 = 0$, in a co-dimension-one subspace of parameter-space defined by:

$$\varepsilon = \frac{1}{4}\left[b_1(2k^2 - 3r_0^2) + 6\alpha_0 c_1(k^2 - r_0^2)\right],$$

where:

$$
\begin{aligned}
k^2 &= \frac{A_0^4}{\alpha_0^2(1 + 4\alpha_0^2)^2(b_1^2 + c_1^2)^2}\left[\, 7\alpha_0(b_1 b_3 - c_1 c_3) + (2\alpha_0^2 - 3)(b_3 c_1 + b_1 c_3)\,\right] \\
&\quad \times \left[\, 3\alpha_0(b_1 b_3 - c_1 c_3) + (2\alpha_0^2 - 1)(b_3 c_1 + b_1 c_3)\,\right], \\
r_0 &= \frac{A_0^2}{\alpha_0(1 + 4\alpha_0^2)(b_1^2 + c_1^2)}\left[\, -3\alpha_0(b_1 b_3 - c_1 c_3) + (1 - 2\alpha_0^2)(b_3 c_1 + b_1 c_3)\,\right].
\end{aligned}
$$

The remaining unknowns are given by:

$$
\begin{aligned}
\omega &= -c_1\alpha_0\theta_0^2 + \frac{1}{4}\left[c_1(2k^2 - 3r_0^2) - 6\alpha_0 b_1(k^2 - r_0^2)\right], \\
\mu_0 &= -\frac{k^2 + r_0^2}{2kr_0}.
\end{aligned}
$$

The asymptotic group velocities $v_{g,\pm}$ in a co-moving frame at $\xi \to \pm\infty$ are given by:

$$v_{g,\pm} = \frac{\partial\Omega}{\partial K_\pm},$$

where the respective asymptotic pulsation and wavenumbers are $\Omega = \omega$ and $K_\pm = \alpha_0(\theta_0 \pm k)$. We obtain:

$$
\begin{aligned}
v_{g,\pm} &= \frac{1}{2\alpha_0}\left(\frac{\partial\omega}{\partial\theta_0} \pm \frac{\partial\omega}{\partial k}\right) \\
&= -\frac{v}{2\alpha_0} \pm \frac{c_1 - 3\alpha_0 b_1}{2\alpha_0}k.
\end{aligned}
$$

In the generic case $b_1 \neq 0$, $v = \theta_0 = 0$, we find

$$v_{g,+} = -v_{g,-} = \frac{c_1 - 3\alpha_0 b_1}{2\alpha_0}k,$$

whose sign can be either positive or negative. These solutions can be either sources or sinks.

4 Conclusion

The search for the functional form of the coherent structures appearing in disordered regimes of extended, non-equilibrium systems can be made systematic by following two basic principles:

- the singularity structure of possible solutions of the relevant DE's must be taken into account at an early stage,

- logarithmic derivatives of entire functions are the elementary units from which exact solutions can be built.

In this work, we followed these principles and derived systematically an Ansatz for coherent structure solutions of the one-dimensional quintic complex Ginzburg-Landau equation. In addition to the already known pulses (Eq. (24)) and fronts (Eq. (25)), we found new source and sink solutions (Eq. (26)). The corresponding computations involved only the simplest possible Ansatz compatible with our framework. We reduced the number of families of entire functions to two, used real coefficients throughout, and chose to define the elementary entire functions from the simplest available ODE. Within these restrictions, exponentials appeared as the simplest building blocks from which exact solutions can be formed. Relaxing these constraints can be expected to yield new - and more complex - solutions, provided that the resulting algebraic computations are not made intractable by increasing the degree of equations to be solved. These less restrictive Ansätze are currently under investigation.

The stability and dynamical relevance of the solutions, especially the new source/sink, remain to be investigated. In [14], a survey of the special analytical, topological and dynamical properties of the highly non-generic fronts (Eq. (25)) was given, and the question of relating these three aspects was raised. These properties are crucial steps in trying to understand the spatiotemporally disordered regimes exhibited by the equation, and deserve closer scrutiny. Another point of interest is the question of a possible role played by these solutions in regions of parameter space out of their domain of existence. There, objects related to these solutions, but out of reach of the simple Ansatz used in this work, may appear as the relevant building blocks in the (chaotic) dynamics.

To conclude, let us stress again the systematic character of the approach taken and the potential extensions of the case treated here, not only to less restrictive Ansätze, but also to other non-linear PDEs of physical interest.

Appendix

We treat here the degenerate case: $\mathcal{I} = b_1 c_5 + b_5 c_1 = 0$. Leading-order analysis then leads to:

$$\alpha_0 = 0, \tag{27}$$
$$A_0^4 = \frac{3}{4} \frac{b_1 b_5 - c_1 c_5}{b_5^2 + c_5^2},$$

whenever the condition (10) applies. This singular behavior is taken into account by writing:

$$A(\xi) = A_0 e^{-i\omega t} \, R(\xi)^{\frac{1}{2}} \, e^{i\Theta(\xi)},$$

where $R(\xi)$ has at most the singular behavior of a pole, and $\Theta(\xi)$ is a regular function. In the spirit of Section 2.3, we write:

$$\begin{cases} R &= r_0 \; + \; \partial_\xi(\mathrm{Log}\psi_1) \; \pm \; \partial_\xi(\mathrm{Log}\psi_2), \\ \partial_\xi \Theta &= \theta_0, \\ \partial_{\xi\xi}\psi_i &= \frac{k^2}{4} \, \psi_i, \quad i = 1, 2. \end{cases} \tag{28}$$

As before, looking for particular solutions of the quintic CGL equation with the restriction $b_1 c_5 + b_5 c_1 = 0$ amounts to solving a system of algebraic equations. Its systematic resolution leads to pulse, front, source and sink solutions. The method is identical to that presented in Section 2.4, and solutions are given within the restrictions of Section 3.

Pulses

The pulse solutions read:

$$A(x,t) = A_0 e^{i[\theta_0 \xi - \omega t]} \left[\frac{k \; \sinh(ka)}{\cosh k(\xi - \xi_0) \; + \; \cosh(ka)} \right]^{\frac{1}{2}} \tag{29}$$

and necessarily respect $\hat{v} = 0$, $b_1 \theta_0 = 0$. We distinguish the two cases:

- **First subcase:** $b_1 \neq 0$. A discrete set of stationary pulses is found in a co-dimension-two subspace of parameter space defined by:

$$
\begin{aligned}
b_1 c_5 + b_5 c_1 &= 0, \\
b_1 c_3 + b_3 c_1 &= 0.
\end{aligned}
$$

Such restrictions include the case of the Real Ginzbug-Landau equation (RGL: $c_1 = c_3 = c_5 = 0$):

$$
\frac{\partial A}{\partial t} = \varepsilon A + b_1 \frac{\partial^2 A}{\partial x^2} - b_3 |A|^2 A - b_5 |A|^4 A. \tag{30}
$$

All parameters are fixed:

$$
\begin{aligned}
v &= 0, \\
\theta_0 &= 0, \\
b_1 k^2 &= -4\varepsilon, \\
b_1 k \mu_0 &= -b_3 A_0^2, \\
b_1 \omega &= \varepsilon c_1.
\end{aligned}
$$

- **Second subcase:** $b_1 = 0$. A two-parameter family of pulses is found for the Quintic-Cubic Schrödinger equation ($c_1 \neq 0$, $c_5 \neq 0$):

$$
-i\frac{\partial A}{\partial t} = c_1 \frac{\partial^2 A}{\partial x^2} + c_3 |A|^2 A + c_5 |A|^4 A. \tag{31}
$$

The free parameters are chosen to be the wavenumber k and velocity v:

$$
\begin{aligned}
2c_1 \theta_0 &= v, \\
4c_1 \omega &= -v^2 - c_1^2 k^2, \\
c_1 k \mu_0 &= A_0^2 c_3.
\end{aligned}
$$

Fronts

Fronts are obtained for $\mu_0 = 0$ and $r_0 = \pm k$:

$$
A(x,t) = k^{\frac{1}{2}} A_0 e^{i[\theta_0 \xi - \omega t]} [\, \tanh k(\xi - \xi_0) \, \pm \, 1 \,]^{\frac{1}{2}}. \tag{32}
$$

The condition $b_1 + ic_1 \neq 0$ leads us to distinguish two cases:

- **First subcase:** $b_1 \neq 0$. A discrete set of fronts is found in a co-dimension-one subspace of parameter space defined by:

$$
b_1 c_5 + b_5 c_1 = 0.
$$

The wavenumber k is a solution of the quadratic equation:

$$a_k k^2 + b_k k + c_k = 0,$$

where:

$$\begin{cases} a_k &=& (3b_1^2 + 4c_1^2), \\ b_k &=& \pm 2A_0^2 \, (b_1 b_3 - 2c_1 c_3), \\ c_k &=& -\varepsilon b_1 + c_3^2 A_0^4. \end{cases}$$

The unknowns θ_0, v and ω are obtained as functions of k:

$$\begin{aligned} b_1 \theta_0 &=& \mp 2c_1 k + A_0^2 c_3, \\ b_1 v &=& \mp 2(b_1^2 + c_1^2)k + 2A_0^2(c_1 c_3 - b_1 b_3), \\ b_1^2 \omega &=& a_\omega k^2 + b_\omega k + c_\omega, \end{aligned}$$

where:

$$\begin{cases} a_\omega &=& -c_1(5b_1^2 + 4c_1^2), \\ b_\omega &=& \pm 2A_0^2(b_1^2 c_3 + 2c_1^2 c_3 - 2b_1 b_3 c_1), \\ c_\omega &=& A_0^4 \, c_3 \, (2b_1 b_3 - c_1 c_3). \end{cases}$$

- **Second subcase:** $b_1 = 0$, $c_1 \neq 0$. A one-parameter family of solutions is found in a co-dimension-three subspace defined by:

$$\begin{aligned} b_1 &=& 0, \\ b_5 &=& 0, \\ \varepsilon c_1 &=& A_0^4 b_3 c_3, \end{aligned}$$

thus including the generalized quintic-cubic NLS equation (Eq. (31)) when $\varepsilon = b_3 = 0$. All coefficients can be expressed as functions of the velocity v:

$$\begin{aligned} 2c_1 \theta_0 &=& v + 2A_0^2 b_3, \\ 4c_1 \omega &=& -v^2 + A_0^4(4b_3^2 - c_3), \\ 2c_1 k &=& \pm A_0^2 c_3. \end{aligned}$$

Sources and sinks

These solutions read:

$$A(x,t) = A_0 e^{i[\theta_0 \xi - \omega t]} \left[r_0 + \frac{k \, \sinh(ka)}{\cosh k(\xi - \xi_0) \, + \, \cosh(ka)} \right]^{\frac{1}{2}}, \qquad (33)$$

and respect $\hat{v} = 0$. We distinguish the two cases:

- **First subcase:** $b_1 \neq 0$. A discrete set of stationary solutions is found in a co-dimension-two subspace of parameter space defined by:

$$\begin{aligned} b_1 c_5 + b_5 c_1 &=& 0, \\ b_1 c_3 + b_3 c_1 &=& 0, \end{aligned}$$

including the RGL equation (Eq. (30)). All parameters are fixed: r_0 is determined by the quadratic equation:

$$3b_1^2 r_0^2 + 4A_0^2 b_3 r_0 - 4\varepsilon = 0,$$

and:

$$
\begin{aligned}
v &= 0, \\
\theta_0 &= 0, \\
b_1 k^2 &= -2A_0^2 b_3 r_0 + 4\varepsilon, \\
b_1 \omega &= \varepsilon c_1, \\
\mu_0 &= -\frac{k^2 + r_0^2}{2k r_0}.
\end{aligned}
$$

The far-field group velocity v_g vanishes, due to an asymptotic stationary wave behavior:

$$\lim_{\xi \to \pm\infty} A(x,t) = A_0 \sqrt{r_0} e^{-i\omega t}.$$

- **Second subcase:** $b_1 = 0$. A two-parameter family of sources or sinks is found for the quintic-cubic NLS equation (31). We choose to use v and r_0 as free parameters ($c_1 \neq 0$):

$$
\begin{aligned}
2c_1 \theta_0 &= v, \\
4c_1 \omega &= -v^2 + 3c_1^2 r_0^2 - 4A_0^2 c_1 c_3 r_0, \\
c_1 k^2 &= 3c_1 r_0^2 - 2A_0^2 c_3 r_0, \\
\mu_0 &= -\frac{k^2 + r_0^2}{2k r_0}.
\end{aligned}
$$

The asymptotic group velocities $v_{g,\pm}$ read:

$$v_{g,+} = -v_{g,-} = \frac{\partial \omega}{\partial \theta_0} = -v.$$

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